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Analytical Solutions for Open-Channel Temperature Response to Unsteady Thermal Discharge and Boundary Heating

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Abstract: Analytical solutions are derived for a one-dimensional model of the bulk temperature response of open-channel flow with unsteady and nonuniform heating at the upstream end, the water surface, and the riverbed. The solutions are explicit formulas comprised of transient terms, which play dominant roles in the upstream region, and equilibrium terms, which determine the temperature far downstream. The applicability of the solutions to practical problems is illustrated for two cases: (1) a stream bounded at its upstream end by a dam and with a midreach inflow; and (2) Boulder Creek, Colo., which is impacted by effluent released from a wastewater treatment plant. The model prediction is in reasonable agreement with gauged data.

Introduction

Although accurate and computationally affordable numerical simulations are becoming the dominant approach for predicting mass and heat transfer in various flows, analytical approaches continue to be useful because of their irreplaceable roles. This technical note presents a model for the bulk temperature of water bodies such as rivers, lakes, and reservoirs that have time-dependent thermal forcing at their upstream ends and water surface, side bank, and bed (hereinafter referred to as lateral boundaries). The model is based on an earlier model described by Edinger et al. (1974) and Jobson and Schoellhamer (1987), and it has been applied in natural stream predictions (Kim and Chapra 1997; Boyd and Kasper 2003), variable reservoir releases (Carron and Rajaram 2001), and river temperature control (Gu et al. 1999). A comprehensive description, including supporting parameters for phenomena such as air–water heat exchange, can be found in Edinger et al. (1974), Jobson and Schoellhamer (1987), and Martin and McCutcheon (1999). Whereas the governing equations of the model are typically solved numerically, analytical solutions are sought in this work.

This study assumes that the flow is fully mixed over its cross section with negligible effects of diffusion and that a one-dimensional model is appropriate. The temperature response is

physically simulated as a series of kinematic waves subject to thermal upstream and lateral boundary conditions, which are either periodic or arbitrary functions of time and space. The resulting governing equation is a first-order linear differential equation that can be analytically solved. The solutions extend the equilibrium solution of Edinger et al. (1974), which is valid far from an upstream region. Analytical solutions have been derived for similar equations (Chatwin 1973; van Genuchten and Alves 1982; Shukla 2002; Weigand 2004). These solutions consider longitudinal diffusion and dispersion of flows with initial conditions and unsteady upstream boundary conditions but ignore time-dependent heating and cooling along the channels.

Conceptual Model and Governing Equations

The one-dimensional governing equation for the bulk temperature variation neglecting diffusion in a semiinfinite stream can be expressed as (Edinger et al. 1974; Jobson and Schoellhamer 1987)

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial s} = K(T_{bd} - T), \quad 0 < s < \infty \tag{1}$$

$$T = T_{up}, \quad s = 0 \tag{2}$$

where t=time; s=distance downstream; T=temperature, T_{up}=temperature at the upstream end; T_{bd}=lateral boundary temperature; u=flow velocity; and K=constant bulk heat transfer coefficient. In order to complete the general problem stated by Eqs. (1) and (2), an initial condition for temperature is needed. This study considers only equilibrium solutions independent of the initial temperature condition, and thus no initial condition is used.

Case 1: Constant Velocity Flow with Sinusoidal Time-Dependent Thermal Discharge and Boundary Heating

Case 1 has the following flow and temperature conditions:

$$u = U \tag{3}$$

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$$T_{up} = \bar{T}_0 + T_0 \sin(\omega_0 t + \alpha) \quad (4)$$

$$T_{bd} = \bar{T}_1 + T_1 \sin \omega_1 t \quad (5)$$

where U=constant velocity; $\bar{T}_0$, T_0, ω_0, and α=average, amplitude, frequency, and phase of the upstream temperature fluctuation, respectively; and $\bar{T}_1$, T_1, and ω_1=average, amplitude, and frequency of temperature at the lateral boundaries, respectively. $\bar{T}_0$, T_0, ω_0, α, $\bar{T}_1$, T_1, and ω_1=constants. Eq. (4) and (5) are frequently used as approximations of measurements (Velz 1970; Edinger et al. 1974). For example, Eq. (4) may model the temperature modulation of water released from detention ponds and reservoirs, whereas Eq. (5) may represent diurnal heating at the water surface. Case 1 is a direct extension of the problem discussed by Edinger et al. (1974), in which Eqs. (1), (3), and (5) were used to describe the thermal response of natural water bodies.

Case 2: Flow with Spatially and Time-Dependent Boundary Heating

Case 2 is an extension of Case 1 but with a spatially variable lateral thermal boundary condition specified as

$$T_{bd} = \bar{T}_1 + \bar{T}'_1 s + (T_1 + T'_1 s)\sin \omega_1 t \quad (6)$$

Eq. (6) can represent temporal and spatial changes in atmospheric temperature, riverbed temperature, groundwater discharge, and snowmelt at higher elevations (Hanrahan 2007; Westhoff et al. 2007).

Case 3: Flow with Spatially Variable Velocity and Time-Dependent Upstream Temperature and Boundary Heating

Case 3 incorporates a spatially variable flow velocity and time-dependent temperature at the upstream and lateral boundaries

$$u = V(s) \quad (7)$$

$$T_{up} = H(t) \quad (8)$$

$$T_{bd} = \bar{T}_1 + \sum_{i=1} T_i \sin(\omega_i t + \beta_i) \quad (9)$$

The right-hand side (RHS) of Eq. (9) defines an arbitrary function of time if the Fourier trigonometric series converges. Therefore, the flow velocity is an arbitrary function of distance, and the inflow and lateral boundary temperatures are arbitrary functions of time.

Solutions and Discussion

Case 1

Eqs. (1)–(5) can be transferred into an initial value problem for a linear, first-order, nonhomogeneous ordinary differential equation,
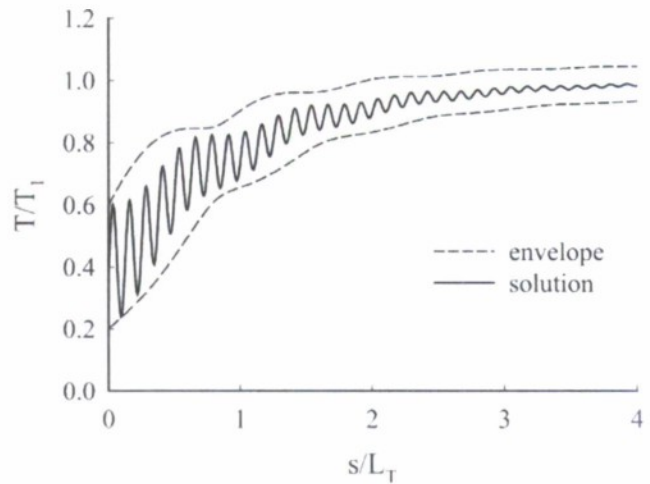

Fig. 1. Base solution of Eq. (10). U=0.1 m s^{-1}, $\bar{T}_1$=50°C, and K=0.04 s^{-1}. $T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t)$, and $T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5)$.

and the solution of Case 1, termed the base solution, can be derived as

$$
\begin{aligned}
T = \bar{T}_1 &+ \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \sin(\omega_1 t - \theta_1) \\
&+ \left[\bar{T}_0 - \bar{T}_1 + T_0 \sin\left\{ \omega_0\left(t - \frac{s}{U} \right) + \alpha \right\} \right. \\
&\left. - \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \sin\left\{ \omega_1\left(t - \frac{s}{U} \right) - \theta_1 \right\} \right] \exp\left\{ -\frac{s}{L_T} \right\} \quad (10)
\end{aligned}
$$

where $\theta_1 = \tan^{-1}(\omega_1/K)$ and $L_T = U/K$. The base solution consists of two parts: (1) the first two terms on the right-hand side of Eq. (10), which reflect the influence of the lateral boundary condition, are the equilibrium solution far downstream as discussed by Edinger et al. (1974); and (2) the exponential term on the RHS, which results from both the upstream and lateral boundary conditions, represents a transient stage decaying with downstream distance. L_T is a characteristic transient length measuring the importance of upstream conditions. The transient stage ends at about $s=3L_T$ at which the transient component will reduce to 5% of its value at upstream.

The temperature solution of Case 1 oscillates with decreasing amplitude until the equilibrium region is reached far downstream (Fig. 1). The solution envelope is also damped downstream, but the damping is not monotonic. This behavior can be explained as follows. Let $\omega_0 = \omega_1$, Eq. (10) becomes

$$T(t,s) = \bar{T}_1 + (\bar{T}_0 - \bar{T}_1)\exp\left\{ -\frac{s}{L_T} \right\} + A \sin(\omega_1 t - \theta_1 + \kappa) \quad (11)$$

where

$$A = \sqrt{\left(B\exp\left\{-\frac{s}{L_T}\right\} - \frac{KT_1}{\sqrt{K^2+\omega_1^2}}\right)^2 + \frac{2BKT_1}{\sqrt{K^2+\omega_1^2}}\left(1+\cos\left(-\frac{s}{L_E}+\theta_1+\gamma\right)\right)\exp\left\{-\frac{s}{L_T}\right\}} \tag{12}$$

$$B = \sqrt{T_0^2 + \frac{K^2T_1^2}{K^2+\omega_1^2} - \frac{2KT_0T_1}{\sqrt{K^2+\omega_1^2}}\cos(\alpha-\theta_1)} \tag{13}$$

$$\gamma = \tan^{-1}\frac{T_0\sin\alpha - (KT_1/\sqrt{K^2+\omega_1^2})\sin\theta_1}{T_0\cos\alpha - (KT_1/\sqrt{K^2+\omega_1^2})\cos\theta_1} \tag{14}$$

$$\kappa = \tan^{-1}\frac{B\exp\{-s/L_T\}\sin(-s/L_E+\theta_1+\gamma)}{KT_1/\sqrt{K^2+\omega_1^2} + B\exp\{-s/L_T\}\cos(-s/L_E+\theta_1+\gamma)} \tag{15}$$

where A=amplitude of the solution envelope and $L_E=U/\omega_1$, being a characteristic envelope length. Eq. (12) determines the envelope wavelength as $2\pi L_E$. The envelope in Fig. 1 has a spatial wavelength of $2\pi L_E/L_T=0.84$.

An interesting situation occurs in applications such as river temperature control when the fluctuating component of the upstream temperature, Eq. (4), is equal to the equilibrium temperature far downstream as given in Eq. (10); $T_0=KT_1/\sqrt{K^2+\omega_1^2}$, $\omega_0=\omega_1$, and $\alpha=-\theta_1$. For example, using the parameters of Fig. 1, the resulting solution reveals that the envelope appears as parallel rather than wavy lines (Fig. 2). The solution (10) now becomes

$$T(t,s) = \bar{T}_1 + \frac{KT_1}{\sqrt{K^2+\omega_1^2}}\sin(\omega_1 t - \theta_1) + [\bar{T}_0 - \bar{T}_1]\exp\left\{-\frac{s}{L_T}\right\} \tag{16}$$

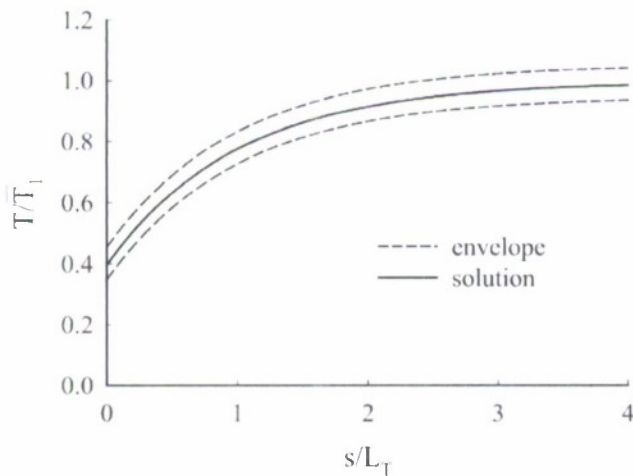The coefficient before the exponential term, the third term on the RHS, is a constant and thus the extremes of the solution will not oscillate with distance, and the temperature envelope is defined by parallel lines.

Case 2

The solution of Case 2 is derived as

$$\begin{aligned}
T(t,s) = {} & \bar{T}_1 + \bar{T}_1's - \frac{U\bar{T}_1'}{K} + \frac{K(T_1+T_1's)}{\sqrt{K^2+\omega_1^2}}\sin(\omega_1 t - \theta_1) \\
& - \frac{KUT_1'}{K^2+\omega_1^2}\sin(\omega_1 t - \theta_1') \\
& + \left[\bar{T}_0 - \bar{T}_1 + \frac{U\bar{T}_1'}{K} + T_0\sin\left\{\omega_0\left(t-\frac{s}{U}\right)+\alpha\right\}\right. \\
& - \frac{KT_1}{\sqrt{K^2+\omega_1^2}}\sin\left\{\omega_1\left(t-\frac{s}{U}\right)-\theta_1\right\} \\
& \left. + \frac{KUT_1'}{K^2+\omega_1^2}\sin\left\{\omega_1\left(t-\frac{s}{U}\right)-\theta_1'\right\}\right]\exp\left\{-\frac{s}{L_T}\right\} \tag{17}
\end{aligned}$$

where $\theta_1'=\tan^{-1}(2\omega_1 K/(K^2-\omega_1^2))$. Eq. (17) with the flow velocity and upstream condition for temperature given in Fig. 1, but with a temporally and spatially variable lateral boundary heating, is plotted in Fig. 3.
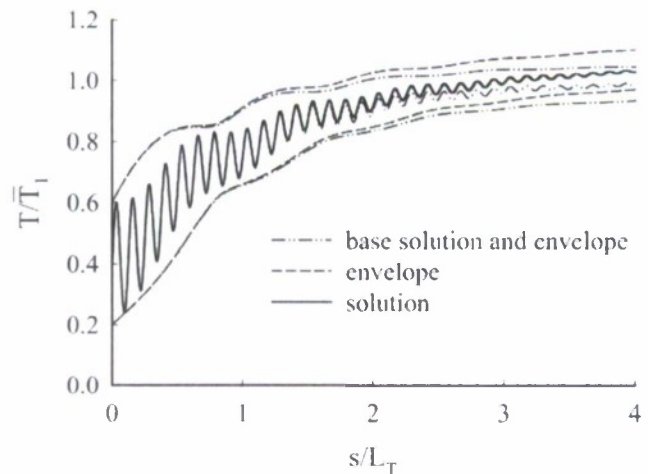

Fig. 2. Base solution of Eq. (10) with the upstream condition determined by the equilibrium temperature far downstream. U $=0.1$ m s^{-1}, $\bar{T}_1=50°$C, and $K=0.04$ s^{-1}. $T_{up}/\bar{T}_1=0.4+0.0326$ $\sin(0.3t-1.438)$, $T_{bd}/\bar{T}_1=1+0.4\sin(0.3t)$.

Fig. 3. Solution of Eq. (17). $U=0.1$ m s^{-1}, $\bar{T}_1=50°$C, and K $=0.04$ s^{-1}. $T_{bd}/\bar{T}_1=1+0.06s+(0.4+0.06s)\sin(0.3t)$, and $T_{up}/\bar{T}_1=0.4$ $+0.2\sin(2t+1.5)$.

Case 3

The solution for Case 3 is obtained as

$$
\begin{aligned}
T = \bar{T}_1 &+ \sum_{i=1} \frac{KT_i}{\sqrt{K^2 + \omega_i^2}} \sin(\omega_i t + \beta_i - \theta_i) \\
&+ \left[H\left(t - \int_0^s \frac{dr}{V(r)} \right) - \bar{T}_1 \right. \\
&\left. - \sum_{i=1} \frac{KT_i}{\sqrt{K^2 + \omega_i^2}} \sin\left\{ \omega_i \left(t - \int_0^s \frac{dr}{V(r)} \right) + \beta_i - \theta_i \right\} \right] \\
&\times \exp\left\{ -K \int_0^s \frac{dr}{V(r)} \right\}
\end{aligned} \qquad (18)
$$

where $\theta_i = \tan^{-1}(\omega_i / K)$. This solution will reduce to the base solution under boundary conditions given by Eqs. (3)–(5). The behavior of Eq. (18) is illustrated in Fig. 4 with three examples that use the parameters given in Fig. 2, but with different flow velocity, upstream temperature, and lateral boundary condition.

Applications

Open-Channel Flow with Time-Dependent Discharge and Surface Heating

Eq. (17) is applied to a river section bounded at its upstream end by a dam with variable discharge water temperature. The river also has diurnal heating, which is approximated by a lateral boundary condition that is not only a sinusoidal function of time, but also spatially variable because of a change in air temperature along its length. The temperature variation at the dam is specified as having the same frequency as that of the surface heating but with a phase delay. The frequency is $1/24$ h^{-1}. The river just downstream from the dam has a mean flow velocity of 1 m s^{-1}, a water depth of 5 m, a mean temperature of $15°C$, and a daily fluctuation of $3°C$. The air has an average temperature of $20°C$ at the dam and $25°C$ at 100 km downstream with a linear increase, an amplitude of $15°C$, and a phase lag of -1.5 rad. According to Edinger et al. (1974), $K = \bar{K}/(\rho C_p h)$, in which ρ = water density $(1{,}000$ kg m$^{-3})$; $\bar{K}$ = air–water exchange coefficient related to factors including wind speed (100 W m^{-2} $°C^{-1}$); C_p = heat capacity of water $(4{,}186$ J kg$^{-1}°C^{-1})$; and h = water depth (5 m). Based on these parameters, $K = 4.8 \times 10^{-6}$ s^{-1}. The analytical solution is plotted in Fig. 5, with the upstream and lateral boundary conditions given in the figure caption. The envelope wavelength $2\pi L_E$ is 86 km. The transient length $3L_T$ is 625 km, suggesting that the river section would not approximate the equilibrium within the distance in the figure.

If a multiple branch flow is considered with a midreach inflow at $s = 50$ km that raises the temperature by $5°C$ and reduces the average flow speed to 0.5 m s^{-1}, then the solution for the reach downstream of s = 50 km section is simply obtained from Eq. (17) by replacing its s with $[s{-}50]$ and its upstream temperature terms (the terms with T_0) with $[H(t) + 5]$. Here, $H(t) = T(t, 50)$, which is
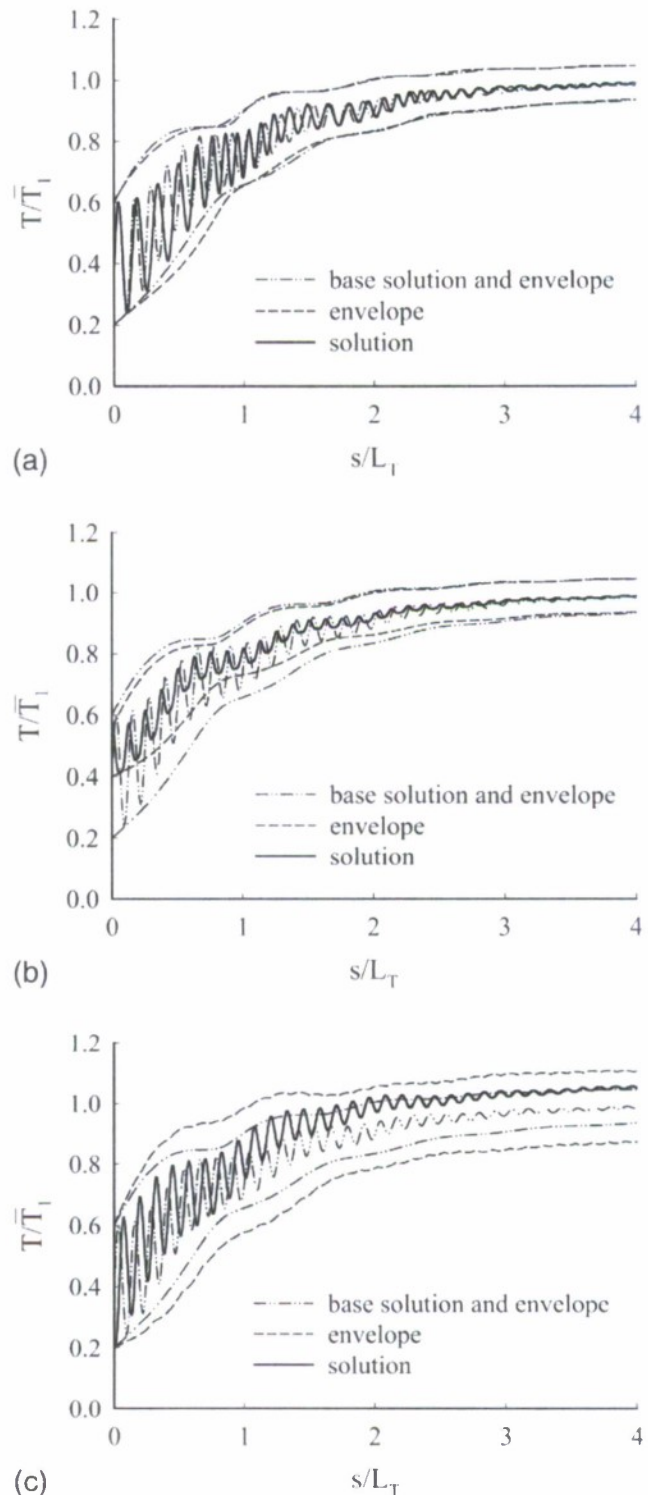
Fig. 4. Solution of Eq. (18). $U = 0.1$ m s^{-1}, $\bar{T}_1 = 50°C$, and $K = 0.04$ s^{-1}: (a) $u(s)/U = 1 + 0.3 \sin(3\pi s/5)$, $T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5)$, and $T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t)$; (b) $u(s)/U = 1$, $T_{up} = 0.4 + 0.04(1 + \sin(2t))^2$, and $T_{bd} = 1 + 0.4 \sin(0.3t)$; and (c) $u(s)/U = 1$, $T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5)$, and $T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t) + 0.2 \sin(0.1t - 1)$.

the solution at $s = 50$ km in case of no inflow and directly determined by Eq. (17). The obtained solution is also shown in Fig. 5, from which it is seen that the solution is significantly changed downstream of the midreach inflow.
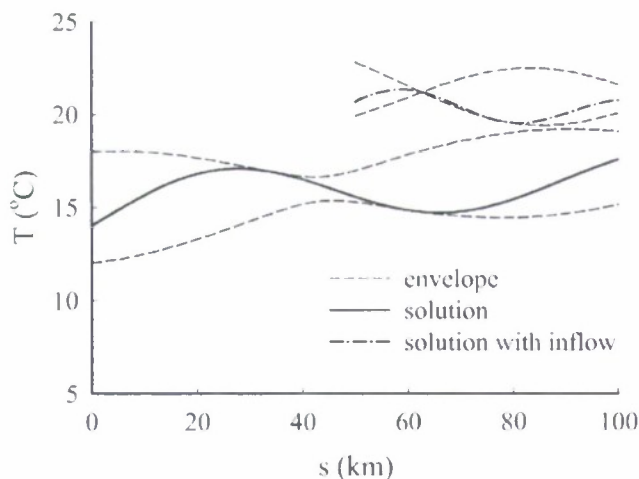
Fig. 5. Temperature response of open-channel flow with a midreach inflow. $U = 1$ m s^{-1}, $T_{up} = 15 + 3 \sin(7.27 \times 10^{-5}t - 1.5)$ °C, $T_{bd} = 20 + 10^{-5}s + 15 \sin(7.27 \times 10^{-5}t)$ °C, $\bar{T}_1' = 10^{-5}$ °C m^{-1}, and $K = 4.8 \times 10^{-6}$ s^{-1}. t and s are in seconds and meters, respectively. An inflow is located at $s = 50$ km, where it raises temperature of the flow by 5 °C and reduces the velocity to $U = 0.5$ m s^{-1} for $s \geqslant 50$ km.

Creek Flow with Effluent from a Wastewater Treatment Plant

The analytical solution of Eq. (18) is applied to Boulder Creek, Colo. which is impacted by effluent discharge from a wastewater treatment plant. A complete description of the flow and a comprehensive 24 h survey can be found in Windell et al. (1988). The creek is 13.7 km long and 0.2–0.6 m deep, the velocity ranges from 0.12 to 0.4 m s^{-1}. The air temperature, wind speed, solar radiation, and other conditions change diurnally. The measurements of creek temperature were made at four stations located 0.6, 5.0, 9.0, and 13.7 km downstream from the plant.

From measurement data (Windell et al. 1988), $u = 0.27$, 3.10, 0.13, and 0.13 m s^{-1} at station 0.6, 5.0, 9.0, and 13.7 km, respectively, $T_{bd} = 15 + 10 \sin(7.27 \times 10^{-5}t)$ °C (t is in seconds); $\bar{K} = 35$ W m^{-2} °C^{-1}; and $h = 0.4$ m. Here T_{bd} is obtained by approximating the effects of air temperature and solar radiation. The velocity is assumed to vary linearly between adjacent stations, and K has been determined in the same way as in the previous case. Letting Station 0.6 km be the upstream end and approximating the observed temperature with $T_{up} = 17 + 3 \sin(7.27 \times 10^{-5}t - 1.5)$ °C [Fig. 6(a)], Eq. (18) is used to predict the temperature at the other stations, which yields Figs. 6(b–d). The analytical solution reasonably estimates the temperature variations at the other stations. The largest difference between the prediction and the measurement occurs at the station furthest downstream [Fig. 6(d)]. Although the temperature predicted by the analytical solution is not as accurate as that from a numerical model by (Kim and Chapra 1997), the errors of the two approaches are of the same magnitude. The results are promising given that the flow and temperature are affected by many factors that the analytical solution excludes, such as the material comprising the creek bed, the thermal interaction between water and sediment, and the three-dimensional character of the flow. To include these factors, a numerical approach has to be adopted (Jobson and Schoellhamer 1987; Kim and Chapra 1997; Tang et al. 2008).
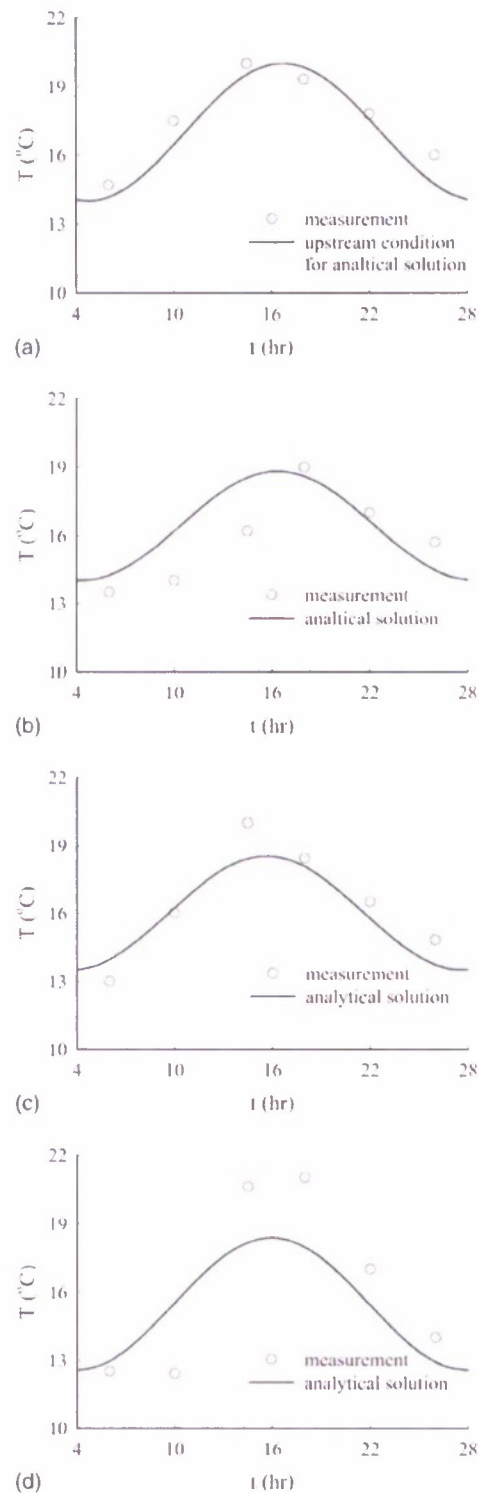

Fig. 6. Temperature response of a creek flow with effluent from a wastewater treatment plant. Define error$= [(1/n)\Sigma_{i=1}^n [(T^i - T_m^i)/T_m^i]^2]^{1/2}$, where $n =$ total number of measurements during the 24 h, and T_m^i and T^i are, respectively, the ith measurement data and the corresponding predicted temperature. (a) Station located at 0.6 km, error$= 0.047$. The solid line is the upstream condition of the analytical model. (b) Station located at 5.0 km, error$= 0.098$. (c) Station located at 9.0 km, error$= 0.056$. (d) Station located at 13 km, error$= 0.13$.

Concluding Remarks

In order to account for flow variability at upstream and lateral boundary conditions, a channel can be divided into several sub-reaches with representative values such as u, T_{bd}, and K; the solutions can then be applied to the individual subreaches. Further, the analytical solutions are explicit formulas consisting of simple functions with a number of controlling parameters. The writers propose as future work to make an Internet-accessible solution tool, such as a small computer code or Excel spreadsheet to allow users to easily set up inputs and obtain solutions.

Acknowledgments

Notation

The following symbols are used in this technical note:

A = amplitude in Eq. (11);
B = parameter in Eq. (12);
C_p = heat capacity;
$H(t)$ = time-dependent upstream condition for temperature;
h = water depth;
i = integers, ≥ 1;
K = coefficient of heat transfer;
$\bar{K}$ = air/water exchange coefficient;
L_E = characteristic envelope length;
L_T = characteristic transient length;
s = distance downstream;
T = mean cross-sectional temperature;
T_{bd} = lateral boundary condition for temperature;
T_i = amplitude of lateral boundary condition temperature due to component i;
T_{up} = inflow temperature at an upstream end;
T_0 = upstream temperature oscillation amplitude;
T_1 = lateral boundary oscillation amplitude for temperature;
T_1' = downstream gradient of the amplitude of the lateral boundary temperature;
$\bar{T}_0$ = average upstream inflow temperature;
$\bar{T}_1$ = mean lateral boundary condition for temperature;
$\bar{T}_1'$ = downstream gradient of mean lateral boundary temperature;
t = time;
U = constant flow velocity;
u = cross-sectional mean flow velocity;
$V(s)$ = variable flow velocity;
α = phase of temperature oscillation at an upstream end;
β_i = phase of lateral boundary condition oscillation for temperature due to component i;
γ, κ = phases in Eqs. (11) and (12);
θ_i = phases in Eq. (18);
θ_1 = phase in Eq. (10);
θ_1' = phase in Eq. (17);
ρ = water density;
ω_i = frequency of lateral boundary temperature oscillation due to component i;
ω_0 = frequency of temperature oscillation at an upstream end; and
ω_1 = frequency of temperature oscillation at a lateral boundary.

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